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This Journal is dedicated to the following aims:

1. THROUGH PUBLISHED STANDARD PAPERS ON THE CULTURE ASPECTS, HNMANISM AND HISTORY OF MATHEMATICS TO DEEPEN AND TO WIDEN PUBLIC INTEREST IN ITS VALUES.
2. TO SUPPLY AN ADDITIONAL MEDIUM FOR THE PUBLICATION OF EXPOSITORY MATHEMATICAL ARTICLES.
3. TO PROMOTE MORE SCIENTIFIC METHODS OF TEACHING MATHEMATICS.
4. TO PUBLISH AND TO DISTRIBUTE TO THE GROUPS MOST INTERESTED HIGH-CLASS PAPERS OF RESEARCH QUOLITY REPRESENTING ALL MATHEMATICAL FIELDS.

W. Paul Webber

At twelve o'clock of the morning of June 26, 1935, W. Paul Webber, for thirteen years a member of the Louisiana State University mathematics staff, walked out of his class room for the last time. Dining at home at his accustomed hour, he lay down for the usual afternoon sleep—a sleep from which he did not awake.

No man has lived who loved truth more, or whose scorn for insincerity and pretense was greater. Not only were his ideals high, but these were maintained with a steadiness that so often marks one whose mathematics has become a well-nigh integral part of his being.

S. T. SANDERS.

Dr. Webber did most of his work in the fields of complex function theory and differential equations. His scholarship in these fields as well as in mathematics in general was very broad. But it was mathematics in relation to human beings which aroused his greatest enthusiasm. He was always interested in new applications of mathematics to human problems. He thought much and deeply on the problems of the mathematics teacher. I acknowledge with gratitude the benefits received from many discussions with him of these problems.

H. L. SMITH.

Dr. Webber and I occupied the same office; we were in close daily contact with each other. Hence, his death was a very personal loss to me.

As a mathematician, he was strong and capable; as a teacher, he was alive and progressive; and as a man of science and philosophy, he was alert. Moreover, he was a man of quiet sense of humor—few people knew this. He loved to study the ways of people, and often expressed himself as "having gotten a kick" out of some certain little incidents that entirely escaped others present. He was, therefore, quite human.

We deeply deplore his untimely death.

IRBY C. NICHOLS.

*Mathematics As An Experimental Science

By H. J. Ettlinger
Professor of Pure Mathematics, The University of Texas

Men endowed with great wisdom have endeavored to give an adequate definition of mathematics. Such men as the English Philosopher and mathematician Bertrand Russell and in this country Cassius Keyser and E. T. Bell have given such definitions, but they are negative rather than positive. Without attempting to follow in their footsteps we may say at the outset that we shall deal with such parts of mathematics as may be included under the description "the science of numbers." Often intelligent persons go at great length to distinguish between the common every-day garden variety of arithmetic and mathematics. I recall a conversation with a visiting professor one summer at the University of Texas, who, I think, hailed from one of the Louisiana state teacher's colleges. I asked the gentleman whether he had occasion to use mathematics in his work as a professor of Pedagogy. He answered, "No." I then asked whether he dealt with correlation coefficients and some of the other techniques of statistical analyses. His response was, "Oh, yes, but that is arithmetic not mathematics." We shall be primarily concerned with those fields in mathematics which may be regarded as the development in direct line of the most elementary notion of number, namely, that connected with the process of counting. Whether these include all of mathematics or not, will not be pertinent.

We shall then start out with the aspects of mathematics which are concerned directly with numbers; and of numbers, we may say, as others have said, that they are the language of science, i.e. of exact knowledge. Very recently I heard a literary exponent of the ultramodern school, Gertrude Stein, use the terminology of mathematics to express the idea of motion and living and growing as opposed to static equilibrium, lifelessness, and rigidity. However, one may view the forms of literary expression introduced by this lady, one readily admits that when she said that the important thing is not that one and one makes two, but it is that one plus one plus one goes on and on and on and on, she struck a note fundamental to mathematics because those aspects of mathematics with which I shall deal are precisely of

* Address delivered at the annual banquet of the Mississippi-Louisiana section of the Mathematical Association of America and the National Council of Mathematics Teachers, and the Louisiana Academy of Sciences, held at Louisiana College, Pineville, La.

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the nature of this process of beginning with one and one and one and going on and on and on. The average conception of the educated layman is that mathematics was created in the beginning full-grown and in perfect form. This idea undoubtedly stems from the fact that in 2500 years we have not had very much to add to the body of Euclidean Geometry as developed by the Egyptians and Greeks. Without question, also, our notion of the exactness of mathematics has part of its origin in this sort of setting. A young lawyer recently made the statement that our laws change every day but that mathematics is the same now, as it was many, many years ago. This young man might have been a good lawyer, but he was certainly a poor mathematician. Clearly mathematics today is not the same as it was in the days of Babylonians, Egyptians, and Greeks, as it was in the days of the Hindus and Arabs, nor is the mathematics of Leibnitz and Newton the same as the mathematics of Planck and Einstein. Just as the counting numbers beginning with one, roll on without ceasing so has the progress and development of mathematics been that of a steady, living growth.

The following four aspects of mathematics as an experimental science will lend themselves to worth-while discussion before this audience, (1) with respect to the methods of mathematics, (2) with respect to the applications ofmathematics, (3) with respect to the teaching of the subject itself, and (4) with respect to the development and growth of the organic body of mathematics. We must recognize that not all these categories are mutually exclusive but that they overlap in a very complete and important fashion. The methods of mathematics are intimately bound up with the kinds of applications we make and the teaching of the subject from the point of view of stimulating classes of young students, is definitely concerned with both of the first two divisions, while the last part which has to do with the growth and development of the subject itself frequently owes its being or existence to the first three. I bear in mind that before me are to be found scientists, interested in as exact a description of their field of knowledge as is possible, as well as teachers of the subject of mathematics and workers in the field. I trust, also, that there may be some among you who have an aesthetic interest in mathematics for its own sake. As citizens we are all profoundly interested in recent projects to revise the curricula, particularly of our junior and senior high schools, with respect to courses of study and to the content of these courses. I anticipate that the usual fulmination and destructive criticism of those untutored in the field of numbers will take place but there is also the hope that bringing these criticisms out into the light

will help in the direction that truth may prevail and the best educational interests of future generations may not be sacrificed. Bearing these various points of view in mind, we shall endeavor to examine the bases for viewing mathematics as an experimental science along these four lines.

We shall begin with certain experimental methods which are used in the field of numbers. The illustrations one might give cover almost the entire field. At the outset let us mention a few which occur in the early stages of working with numbers. A large class of these examples come to mind as soon as we mention the process known as "trial and error." Clearly this is of the very essence of the experimental method. In this connection I have always stressed with my own students the importance of good guessing in mathematics, provided the guess will stand up. In other words would not a method whereby one can write down a solution in the briefest interval of time and with the smallest amount of expenditure of mental energy, be generally useful. In short, if one can solve a problem by inspection, by all means do so, but be sure to check the keen glance of your eagle eye. This same method is known in some branches of numbers as the method of successive approximation. A simple example of this may be had in connection with our definition and computation of certain irrational numbers. The counting numbers may be added and multiplied and the result is always a counting number. But when we divide one counting number by another number we do not always obtain a counting number as the result. As a matter of fact this is where our notion of fractions or ratio numbers (also called rational) enters. We may now divide one number by another number, add numbers, or multiply numbers, and the result is a number in the sense of rational or "ratio" numbers. We step outside of this class of "ratio" numbers when we ask ourselves the question, what number multiplied by itself will give two as the result? The answer is, there is no such rational number. Such a number is outside of the class of rational numbers, in other words, is irrational. How can we find irrational numbers? If we will assume for the moment that we have forgotten, or that we never knew the ordinary method of finding the square root, but that we do know how to multiply and divide two rational or decimal numbers we can find the number, the square root of two, by the following process. We know that our desired number is larger than one and smaller than two, since one multiplied by itself equals one, and two multiplied by itself equals 4. Let us split the difference between one and two and try 1.5. We now divide 1.5 into the number two and obtain as our result the number 4/3 or taking a smaller number 1.3. Clearly

1.3 is too small since 1.3 multiplied by itself is 1.69, and 1.5 is too large since 1.5 multiplied by itself is 2.25. We then split the difference between 1.5 and 1.3 and obtain 1.4. We divide two by 1.4 and obtain 1.43. By multiplying again we see that 1.4 is too small, that 1.42 is too large, we then average and obtain 1.41. By continuing this process we may obtain two numbers that will be alike except that they differ by one in any pre-assigned decimal place between which two numbers the number, the square root of two, is to be found. This is called in mathematics finding the square root of two to any desired degree of approximation. In fact, it illustrates in this method the very definition of the irrational number, the square root of two itself by means of either of the sets of numbers obtained by this process one, 1.4, 1.41, and on and on and on or 2, 1.5, 1.42, and on and on and on.

Without going at length into the method we may see that by this same process of "trial and error" one may obtain the value of the famous irrational number Π representing it as the number obtained by dividing the number of square units contained in the area of a circle by the number of square units in the square of the radius of the circle. Such a process of determining the value of Π experimentally is embodied in counting the number of square units lying entirely within a circle compared with the number of square units which lie entirely inside and those any interior points of which lie inside. The process in this case continues by taking successively smaller unit squares. One might multiply the number of examples of this type indefinitely, taking it on into more advanced number relationships, such as are embodied in what is called the numerical solution of equations in various fields together with the determination of sets of numbers which occur in what is known as differential equations. The essence of this method of "trial and error" is always of importance in connection with the use of numerical tables either of the ordinary arithmetic variety, squares and square roots, cubes and cube roots, or of the more sophisticated variety, such as logarithms or exponents, angle or circle numbers and probability numbers. In this connection we recognize that numbers obtained by this method are in constant use by engineers and others who use what is known as the series expansion of numbers. In all these connections the question of limit of accuracy enters and no experimental method in mathematics or in any other science can be regarded as satisfactory unless a test of the limit of accuracy can be made. I recall a statement by a well known professor of engineering that he preferred the series form of certain numbers instead of obtaining their values directly from tables. Because, he said, "You only need to use the first few terms of the series." His remark, however, showed

lack of understanding of the notion of limit of error or limit of accuracy in connection with the use of series.

Another illustration of the experimental method is the determination of a curve of prescribed type through a definite set of points. This is called in mathematics, the method of interpolation. I shall not go further into this type of example. I point out also that mechanical and graphical mehods in mathematics are essentially of the experimental type and widely used, and in many fields solutions or answers of a specified type are frequently introduced, on trial, to be verified and checked later. A rather striking example which did not check out was the classic assumption that the motion of a perfect fluid could be used to serve as an approximation for the motion of a fluid with small coefficient of internal friction.

With respect to the second experimental aspect of mathematics, namely the application of mathematics, you must have noticed that a number of the examples given under section (1), could and would be regarded by some as applications of mathematics. However, there is to be added with respect to applications several additional ideas which enter from the experimental point of view. In endeavoring to set up what is known as a frame of reference in terms of which one may express quantitatively the relationships involved in a given event or situation there has been a constant and unending process of searching for better and better setups. We speak in a general way of a coordinate system as a set of numbers which describe in a definite manner a given situation. This may be an astronomical one or a physical one or a chemical one or even a biological one. The experimental method in mathematics is illustrated by the constant search for better and better frames of reference. One may cite the fact that in physics we are constantly using the method known as "taking account of first-order effects" which results in yielding first degree relations or what is known as linear equations. I content myself by giving two such examples, namely, the Ptolemaic system and the Copernican in astronomy and the Newtonian and Einsteinian space-time systems in physics. One also notes the experimental methods in applications of mathematics both in the field and outside, with respect to our whole conception of the nature of the universe. Formerly it was thought that nature abhorred vacua and discontinuities. Our present notions show distinct leaning toward vacuity and discontinuity. In this connection we should mention the use of the continuous vibrating string to replace what we actually believe at the present time to be a vibrating string of beads. It is rather deeply significant, and tremendously fascinating to recognize, that the connections between the con-

tinuous and discontinuous, between the vacuum and the all-pervading ether aspects represented by the contest between modern quantum method versus continuous waves in our description of material objects has its roots deep within the experimental aspects of mathematics. It is also possible that the presence of unique solutions in physics in place of band solutions with concomitant uncertainties may be the situation in nature rather than the determinate unique result assumed at present.

With respect to the teaching of mathematics may I state the following important bearings from an experience of twenty-two years of initiating college freshmen and others into the mysteries of the subject. To some, acquiring mathematics is a matter of memory. I recall an extreme case of a teacher of mine in trigonometry who required knowing 146 formulas of the subject by memory and the quoting of them by the numbers. Memory work is not a necessary part of mathematics, rather should it be the doing and in this doing the experimental side of mathematics looms significant. To others, learning mathematics is entirely the acquiring of skill in manipulating mysterious symbols which mean little or nothing to the operator. It is true that the technique in handling given kinds of numbers must be acquired and facility developed, but emphasis on the experimental method will give far more meaning and life to the subject. I learned with some surprise that to certain professors of English and professors of Philosophy on our Texas faculty mathematics meant nothing but memory work and mysterious manipulation. In this connection, also, I should like to mention in passing that I have a definite feeling that elementary plane geometry should be taught more as an experimental course with drawings and diagrams and constructions than as a purely demonstrative one.

The entire history of mathematics, of its development, of the growth of the concept of numbers is a living example of the experimental method in mathematics. It is true that we have not changed very much of the older Greek Euclidean geometry but we have added new approximations or new set-ups called non-Euclidean geometry. In the history of mathematics there has also played a very important role of what is often called the heuristic or intuitional method. This experimental aspect of mathematics has been fundamental in the life of the subject itself. All in all, we may say that there is a plentitude of solid ground on which to take the stand that mathematics is not alone an exact science but in the literal sense of the word many of its aspects are directly experimental.

Early American Arithmetics

By E. R. Sleight,
Albion College, Michigan

The first arithmetic to be written by an American and published in this country appeared anonymously in 1729. It is generally believed, however, that Isaac Greenwood, Professor of Mathematics in Harvard University was the author. This text was not widely used, and there seems to be only three copies now in existence—two in the library at Harvard and one in the Congressional library. In the preface we find the following statement: "The Author's Design in this Treatise is to give a very concise account of such Rules as are of the easiest practice in all the Parts of Vulgar and Decimal Arithmetic".

Perhaps the most striking reason for believing that the Greenwood Arithmetic was little known lies in the fact that Pike's Arithmetic, which appeared better than half a century later, is very generally referred to as the first American work of its kind. Previous to the Revolutionary War, English texts were used in the schools of the colonies, but inspired by the results of that great struggle, there began a period of unusual activity in writing school books. Not the least famous of all of these was "A New and Complete System of Arithmetic Composed for Use of Citizens of the United States" written by Nicholas Pike in 1788. In 1793 he published a smaller work, which he designated as "An Abridgement of the New and Complete Arithmetic composed for the use, and Adapted to the Commerce of the Citizens of the United States for Use of Schools, and will be found to be an Easy and Sure Guide to the Scholar". The abridged text omitted the subjects of logarithms, trigonometry, algebra and conic sections—subjects which were treated so briefly as to possess little value. However, in later editions of the abridged text we do find a very simple treatment of logarithms.

Pike secured encouraging recommendations from the presidents and several of the professors of Yale, Harvard, and Dartmouth, as well as the Governor of Massachusetts. Even Washington acknowledged the receipt of a copy of the book as follows: "Its merits being established by competent judges, I flatter myself that the idea of its being an American production, and the first of its kind which has appeared, will induce every patriotic character to give it all the countenance in his power."

In contrasting Pike's arithmetic with the English texts used by the colonists one is impressed with the lack of "Pleasant and Diverting Questions" which characterized the English texts. Everyone has heard about the farmer who tried to cross a river with a fox, a goose and a sack of corn. Then there is that other equally well known problem relating to three men and their three jealous wives crossing, perhaps, that same river. These and many others appeared in those texts which had been imported from England, but we find them entirely ignored in this First American Arithmetic.

The fifth edition of the abridged text was published in 1804 by J. J. Buckingham in Boston, and it is this particular edition to which I have had access. Arithmetic is here defined as "The Science of Computing Numbers, and is comprised under five principles or fundamental Rules, viz., Notation or Numeration, Addition, Subtraction, Multiplication, and Division". The first thirty-one pages are devoted entirely to these five fundamental processes. Here, as in the entire text, complete rules are enunciated for every type of problem. Also, one is impressed with the variety of tables found in the text as "aids to the scholar". In these first thirty-one pages are found addition and subtraction tables, as well as tables for multiplication and division.

Life may have been very simple in some ways during this period, but in others it was decidedly complex. Under the general topic of Compound Addition we find tables for money in terms of the English units, farthings, pence, shillings and pounds; tables for cloth measure in which are found such terms as Nails (2½"), and 3 varieties of the ell—English (45"), Flemish (27"), and the French (54"). Under long measures is the barley corn (1"), the furlong (1/8 of a mile), and the pole (1/40 of a furlong). Wine was sold according to one system while ale and beer were sold by another. In these two systems we find uch terms as Anchor of Brandy, a Pipe of Wine and a Butt of Ale or Beer. Honey and oil were sold by wine measure, while milk was sold by beer measure.

Then when we come to investigate the question of currencies we are even more confused, for we find "Rules and tables for reducing the Federal Coin, and the currencies of the several States; also English, Irish, Canada, Nova Scotia, Livres Tournois, and Spanish milled Dollars, each to the par of all the others". In traveling from the United States into Mexico or Canada we find some confusion in regard to exchange and in adjusting ourselves to the monetary system of these countries. What must have been the difficulties in those early days when each state had its own system, and in each state it was necessary to keep in mind the various coins of foreign countries which

were then in circulation? According to one table £1 in Massachusetts was equivalent to $3.33 in federal coin, while in New York its value was £1 6s, in Pennsylvania £1 5s, and in Georgia 15s 6 6/9 d. Congress was having its troubles with the monetary systems as seen by the fact that one of the tables is headed by the following: "Table of English and Portuguese gold in Dollars, Cents, and Mills, agreeable to an act of Congress, passed in November 1795", while another table has for its heading "Weights of French and Spanish Gold in Dollars, Cents, and Mills, agreeable to the Act aforesaid".

We find that the subject of money is not confined to the mere use of it for buying and selling with its attendant exchange values, but also, considerable space is given to such topics as Arbitration of Exchanges, Single and Double Fellowsip, and Tare and Tret which "are practical rules for deducting certain allowances made by merchants and tradesmen in selling the goods by weight". Also, we find simple interest, simple interest by decimals, simple interest in federal money, discount, barter, loss and gain, policies of insurance, annuities, commission and brokerage. In fact 150 pages of a total of 352 in the entire book are given to considerations of money in one form or another.

No one type of operation is more often used than the Rule of Three, which "teacheth, by having three numbers given, to find a fourth, that shall have the same proportion to the third as the second has to the first: If *more* require *more* or *less* require *less*, the question belongs to the Rule of Three direct. But if *more* require *less* or *less* require *more*, it belongs to the Rule of Three Inversed". In a footnote we find that "this rule, on account of its great and extensive usefulness, is sometimes called the Golden Rule of Proportion; for on a proper application of it, the whole benefits of Arithmetic, as well as every mathematical inquiry, depends". The text itself is not divided into chapters, but rather into topics of which 7 of a total of 84 are devoted entirely to this Golden Rule, while rules for solving many other topics are given in terms of this process.

No textbook on Arithmetic written during this period would be complete without the subject of alligation, which "is the method of mixing two or more samples of different qualities so that the compound may be of a mean or middle quality. It consists of two kinds, viz., Alligation Medial and Alligation Alternate. Alligation Medial is when the quantities of several things are given to find the mean price of the mixture compounded of those things. Alligation Alternate is the method of finding what quantity of each of the ingredients whose rates are given will compose a mixture of a given rate".

Involution and evolution are given some space, even to the extent of indicating "a general rule for extracting the roots of all powers".

The author recognizes the existence of arithmetic and geometric progressions, permutations and combinations, and logarithms, and gives each a bit of space in his book. These topics are treated in a very simple manner, and are of very little value. In reviewing the book one is greatly impressed with its merits as a class room text but from the standpoint of its usefulness "to the Commerce of the Citizens", it must have been a very much needed contribution.

Pike's confidence in his work is shown by the fact that he registered as an author in Pennsylvania, South Carolina, Massachusetts and New York, this being necessary to serve as a copyright notice. His confidence was fully justified, as the original work went through eight editions, and the abridgement continued to appear until 1830. He was the first American to attain wide popularity in the field of textbooks, and in this way he made a lasting contribution to American Education.

Bibliography

Nicholas Pike: *Complete System of Arithmetic*, second edition. Published in 1797.

Nicholas Pike: *Abridged System of Arithmetic*, fifth edition. Published in 1804.

Cajori: *Mathematics in the United States.*

Cajori: *History of Elementary Mathematics.*

At the college level the student encounters another difficulty in his pursuit of mathematics. A new set of ideas at the beginning of higher analysis ordinarily known as calculus must be assimilated. Here he must interpret the directly incomprehensible in terms of his previous experience. The derivative, the differential and the integral are not simple ideas to the beginner. He must have a background for their assimilation. But when once understood they become most powerful instruments for study of the laws of nature. The perfection of theoretical mechanics as a servant of engineering is a well known example. In mechanics the general methods of Lagrange and Hamilton are of the highest importance in the study of the recent developments in theoretical physics. Such theories and methods are attained as climactic results of long periods of investigation and generalization. Long periods of study of associated ideas are necessary to such grand flights of generalized methods. —W. Paul Webber

Some General Formulas Suggested by An Elementary Identity

By FREDERICK H. HODGE
Purdue University

We start with the identity

$$\cos 20° \cos 40° \cos 80° = \tfrac{1}{8}$$

which is to be proved. Perhaps the first method of proof that suggests itself is to put

$$(\cos 20° \cos 40°) \cos 80° = \tfrac{1}{2} (\cos 60° + \cos 20°) \cos 80°$$

$$= \tfrac{1}{2}(\tfrac{1}{2} + \cos 20°) \cos 80° = \tfrac{1}{2} [\tfrac{1}{2} \cos 80° + \tfrac{1}{2} (\cos 100° + \cos 60°)]$$

$$= \tfrac{1}{2}[\tfrac{1}{2} \cos 80° - \tfrac{1}{2} \cos 80° + \tfrac{1}{4}] = \tfrac{1}{8}$$

A somewhat more general result is obtained by starting with $\cos \theta \cos(60° - \theta)\cos(60° + \theta)$. We then get

$$\cos \theta \cos(60° - \theta) \cos(60° + \theta)$$

$$= \cos \theta(\cos 60° \cos \theta + \sin 60° \sin \theta)(\cos 60° \cos \theta - \sin 60° \sin \theta)$$

$$= \cos \theta(\cos^2 60° \cos^2 \theta - \sin^2 60° \sin^2 \theta) = \cos \theta \frac{(\cos^2 \theta - 3 \sin^2 \theta)}{4}$$

$$= \cos \theta \frac{(\cos^2 \theta - 3 + 3 \cos^2 \theta)}{4} = \frac{4 \cos^3 \theta - 3 \cos \theta}{4} = \frac{\cos 3\theta}{4}$$

We thus have

$$(1) \qquad \cos \theta \cos(60° - \theta)\cos(60° + \theta) = \frac{\cos 3\theta}{4}$$

If $\theta = 20°$ this gives

$$\cos 20° \cos 40° \cos 80° = \frac{\cos 3(20°)}{4} = \frac{\cos 60°}{4} = \tfrac{1}{8}$$

We can get an indefinite number of identities by giving θ different values in (1). For example, if $\theta=10°$ we get $\cos 10° \cos 50° \cos 70° = \sqrt{3}/8$. These identities may be verified directly.

An analogous formula for $\sin\theta \sin(60°-\theta) . \sin(60°+\theta)$ is readily obtained by a similar method

$$\sin\theta \sin(60°-\theta) . \sin(60°+\theta) = \sin\theta(\sin 60° \cos\theta - \cos 60° \sin\theta)$$

$$(\sin 60° \cos\theta + \cos 60° \sin\theta)$$

$$= \sin\theta(\sin^2 60° \cos^2\theta - \cos^2 60° \sin^2\theta) = \sin\theta \left\{\frac{3\cos^2\theta - \sin^2\theta}{4}\right\}$$

$$= \sin\theta \left\{\frac{3 - 3\sin^2\theta - \sin^2\theta}{4}\right\} = \frac{3\sin\theta - 4\sin^3\theta}{4} = \frac{\sin 3\theta}{4}$$

Thus we have

$$(2) \qquad \sin\theta . \sin(60°-\theta) . \sin(60°+\theta) = \frac{\sin 3\theta}{4}$$

If we put $\theta=10°$ this gives us the identity

$$\sin 10° \sin 50° \sin 70° = \frac{\sin 30°}{4} = \frac{1}{8}$$

This again, together with an indefinite number of other identities which can be obtained from this formula, can be verified directly. It is possible to prove our original identity algebraically.

Let $x=\cos 20°$. Then we get, from the formula for $\cos 3x$,

$$4x^3 - 3x = 4\cos^3 20° - 3\cos 20° = \cos 3(20°) = \tfrac{1}{2} \;.\; \therefore\; 8x^3 = 6x+1$$

We now have

$$\cos 20° \cos 40° \cos 80°$$

$$= x(2x^2-1)\,[2(2x^2-1)^2-1] = \tfrac{1}{4}\,(8x^3-4x)(8x^4-8x^2+1)$$

If we replace $8x^3$ by $6x+1$ and $8x^4$ by $6x^2+x$ this becomes

$$\tfrac{1}{4}(6x+1-4x)(6x^2+x-8x^2+1)$$

$$= \tfrac{1}{4}(2x+1)(-2x^2+x+1) = \tfrac{1}{4}(-4x^3+3x+1)$$

$$= \tfrac{1}{8}(-8x^3+6x+2) = \tfrac{1}{8}(-6x-1+6x+2) = \tfrac{1}{8}$$

and again the original identity is proved.

We will now return to our first method of proof and apply it to the expression $\cos\theta\cos 2\theta\cos 4\theta$. We get

$$(\cos\theta\cos 2\theta)\cos 4\theta$$

$$=\frac{\cos\theta+\cos 3\theta}{2}(\cos 4\theta)=\frac{\cos 5\theta+\cos 3\theta+\cos 7\theta+\cos\theta}{4}$$

We thus have

$$\cos\theta\cos 2\theta\cos 4\theta=\frac{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}{4} \tag{3}$$

If $\theta=20°$ this gives

$$\cos 20° \,.\, \cos 40° \,.\, \cos 80°$$

$$=\frac{\cos 20°+\cos 60°+\cos 100°+\cos 140°}{4}$$

$$=\frac{\cos 20°+\frac{1}{2}+2\cos 120°\cos 20°}{4}=\frac{\cos 20°+\frac{1}{2}-\cos 20°}{4}=\frac{1}{8}$$

and our original equation is again proved.

Equation (3) suggests the possibility of a general formula for $\cos\theta \,.\, \cos 2\theta \,.\, \cos(2^2\theta) \,.\,.\,.\,.\, \cos(2^{n-1}\theta)$ of the form

$$\cos\theta \,.\, \cos 2\theta \,.\,.\,.\,.\,.\,.\, \cos 2^{n-1}\theta$$

$$=\frac{\cos\theta+\cos 3\theta+\,.\,.\,.\,.\,.\,.\,+\cos(2^n-1)\theta}{2^{n-1}}$$

We prove this formula by mathematical induction. We assume it to be true for $n=\gamma$ and multiply both sides of the equation by $\cos 2^\gamma\theta$ and thus we get

$$[\cos\theta\cos 2\theta \,.\,.\,.\,.\,.\,.\, \cos(2^\gamma-1)\theta]\cos 2^\gamma\theta$$

$$=\left[\frac{\cos\theta+\cos 3\theta+\,.\,.\,.\,+\cos(2^\gamma-1)\theta}{2^{\gamma-1}}\right]\cos 2^\gamma\theta$$

$$=\cos(2^\gamma+1)\theta+\cos(2^\gamma-1)\theta+\cos(2^\gamma+3)\theta+\cos(2^\gamma-3)\theta+\,.\,.\,.$$
$$+\cos(2^\gamma+2^\gamma-1)\theta+\cos(2^\gamma-2^\gamma+1)\theta/2^\gamma$$

$$=\frac{\cos\theta+\cos 3\theta+\ldots+\cos(2^{\gamma+1}-1)\theta}{2^{\gamma}}$$

Hence, if the formula holds for $n=\gamma$ it will hold for $n=\gamma+1$. But we have seen that it holds for $n=3$. Hence it holds for all positive integral values of n.

Another method of proof is to write

$$\cos\theta\cos 2\theta\cos 4\theta$$

$$=\frac{2\sin\theta\cos\theta\cos 2\theta\cos 4\theta}{2\sin\theta}=\frac{\sin 2\theta\cos 2\theta\cos 4\theta}{2\sin\theta}$$

$$=\frac{2\sin 2\theta\cos 2\theta\cos 4\theta}{4\sin\theta}=\frac{\sin 4\theta\cos 4\theta}{4\sin\theta}$$

$$=\frac{2\sin 4\theta\cos 4\theta}{8\sin\theta}=\frac{\sin 8\theta}{8\sin\theta}$$

Thus we have

$$\text{(5)}\qquad \cos\theta\cos 2\theta\cos 4\theta=\frac{\sin 8\theta}{8\sin\theta}$$

If $\theta=20°$ this becomes

$$\cos 20°\cos 40°\cos 80°=\frac{\sin 160°}{8\sin 20°}=\frac{\sin 20°}{8\sin 20°}=\tfrac{1}{8}$$

and again we have proved our original identity.

Equation (5) suggests a general formula of the form

$$\text{(6)}\qquad \cos\theta\,.\,\cos 2\theta\,\ldots\ldots\cos(2^{n-1}\theta)=\frac{\sin(2^n\theta}{2^n\sin\theta}.$$

This can be easily proved from (5) by induction. Comparing (4) and (6) we get another general formula.

$$\text{(7)}\qquad \cos\theta+\cos 3\theta+\ldots+\cos(2^n-1)\theta=\frac{\sin(2^n\theta)}{2\sin\theta}.$$

If $\theta \neq 0$ $\sin(2^n\theta)/2 \sin \theta$ is a finite number since we have a fraction with a finite number as numerator and a denominator different from zero. Hence we conclude that the sum of the cosines of the odd multiples of any angle θ up to $(2^n-1)\theta$ is finite however large n may be. If $\theta=0$ each term on the left side of the equation is 1 and when we note that the number of odd integers up to and including 2^n-1 is 2^{n-1} and also that $\lim_{\theta \to 0} \sin 2^n\theta/2 \sin\theta$ is also 2^{n-1} we see that our formula is verified for $\theta=0$.

Mathematicians are just as exacting with their technique of execution as any poet or artist is. They are constantly preoccupied with the elegance of their proofs or of the solution of their problems. Any mathematician will immediately assign to the scrapheap any of his proofs if he can think of another way to get the results with less apparent effort, with the accent on the word "apparent." He does not hesitate to spend a great deal of extra time on the solutions he has already, if he has any inkling that he may abbreviate or simplify these solutions. And when he succeeds, when he has found this simplicity he has the esthetic satisfaction of having brought forth an elegant solution. Nor is this effort limited to the individual. Mathematicians as a profession are always at work making the exposition of their science esthetically more satisfying. The success they achieve in this labor is often remarkable. Some of the results which the original discoverers have obtained in a most laborious way, making use of the most advanced and complicated branches of the science, may become within a generation or two, very simple, very elegant, and based on almost elementary considerations. The beauty of this new way of execution becomes then the joy and pride of the profession.

—From Nathan Altshiller-Court's "Art and Mathematics," as published in Scripta Mathematica, Vol. III, No. 2, April, 1935.

"It would seem that enough has been said to show that mathematics pervades and affects the affairs of men in all their varied activities as certainly as the sunshine pervades the earth and supports the life of all animate things. The importance of mathematics is gradually seeping through to the public, as is evidenced by the many research laboratories now financed by the commercial concerns and by the many appeals of smaller concerns for help in their efficiency programs." —From H. E. Slaught's "*Mathematics and Sunshine*" in *The Mathematics Teacher*.

Successive Smoothings of a Time Series

By L. J. Adams
Santa Monica, California

Investigators in many fields of endeavor find it expeditious, and sometimes necessary, to display accumulated data in the form of broken-line graphs. Often they find it useful to change the broken line into a smooth curve which will show at a glance the general trend of the data. In the case of discrete values the line segments joining successive points have a tendency to obscure rather than to emphasize this trend.

The writer presents the following procedure for successive smoothings of a time series, based upon the idea of centroids:

Consider the set of discrete values $P_1, P_2, \ldots, P_n$ represented by points on a set of rectangular Cartesian coordinate axes. Find the centroid of the triangle $P_1 P_2 P_3$. An easy way to do this is to locate the intersection of any two medians of the triangle. An alternative method is to find the arithmetic average of the abscissa of the three points and likewise for the ordinates. The results will be the abscissa and the ordinate of the centroid. Call this point S_1. It will be a point on the smooth curve sought. Repeat the process for the triangles $P_2 P_3 P_4$, $P_3 P_4 P_5, \ldots, P_{n-2} P_{n-1} P_n$. Draw a smooth curve through the points $S_1, S_2, S_3, \ldots, S_{n-2}$. This smooth curve will indicate the general trend of the data.

A second smoothing, showing in still more general way the trend of the data, may be obtained by finding the centroids of groups of four points. This is done by dividing the quadrilateral into two triangles by a diagonal, and finding their centroids. Call these centroids C_1 and C_2. Then use the other diagonal of the quadrilateral and call the centroids of these two triangles C_3 and C_4. The intersection of the lines C_1 C_2 and C_3 3_4 will be the centroid of the four points, and will be the first point on the smooth curve sought.

The number of smoothings made must be determined by the investigator, who will base his decision upon the type of data at hand and the use to which the smooth curve is to be put.

If the following points be plotted on ordinary graph paper, the method will be apparent:

Original Data (P_i)	First Smoothing (S_i)	Second Smoothing (S_i^1)
1, 6		
2, 3	2, $4\frac{1}{3}$	
3, 4	$3\frac{1}{3}$, 5	$3\frac{1}{3}$, $5\frac{1}{3}$
5, 8	5, $6\frac{1}{3}$	$4\frac{3}{4}$, $6\frac{1}{4}$
7, 7	$6\frac{2}{3}$, 9	$5^5/_6$, $7\frac{3}{4}$
8, 12	$8\frac{1}{3}$, $9\frac{2}{3}$	$7^7/_{12}$, $9^2/_5$
10, 10	11, 11	$9\frac{1}{4}$, $10\frac{1}{4}$
15, 11	15, 9	$13\frac{3}{4}$, $9^3/_5$
20, 6		

Note that the abscissa of S_1 is the average of the abscissas of P_1, P_2, and P_3, and that the ordinate of S_1 is the average of the ordinates of P_1, P_2, and P_3.

Each successive smoothing begins farther to the right and ends before the last point of the previous curve. This is in agreement with the caution exercised by statisticians in trends during recent years. The given data do not have to be evenly spaced. The points of each smoothing are farther apart than those of the preceding curve. The second smoothing can be accomplished best by the graphical method described above.

Further smoothing can be made by considering groups of five, six, etc. points at a time and finding their centroids.

Significant Figures

By Carroll W. Griffin
Vassar College

There are several reasons for emphasizing the importance of the proper retention of significant figures in applied mathematics. The number of such figures employed to express a certain quantity should, first of all, tell something of the precision involved. When more digits are used than are warranted the implication becomes audacious, since the user gives the impression of exceeding the limitations of refinement of the apparatus from which observations have been made. When a lesser number of digits are employed than are warranted the user is denying himself his just rights: such indicates carelessness on his part or ignorance of his rights. Thus in either case the impression created might be as much in error as a result which, instead of being at fault regarding the indicated *precision*, was really *inaccurate*.

The habit of disregarding significant figures, once developed, is not easily corrected. Few of us but have seen those who should know better, or who do know better but have become careless, list several pieces of data, all relating to the same type of observation, yet with part expressed, say, to tenths and part expressed to hundredths or even thousandths! In popular publications, we frequently see such examples and sometimes in more pretentious literature. Often no harm is done by such disregard as the reader frequently can correctly guess the true state of affairs; however, in scientific records no guessing should be tolerated if it can possibly be avoided, and certainly there is little elegance in slipshod methods.

In order to clarify the subject a few common notations in usage are stated.

A number is an expression of quantity.

A figure or digit is any of the characters: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.

A significant figure is one which indicates the magnitude of the quantity in the place in which it stands. A number should express the measure of a quantity to an extent that does not go beyond the first uncertain digit. In the number 123 the digits state that there are one hundred, two tens, and three units. They are all significant. The zero is employed in two ways, either as a significant figure or only to locate the decimal point. If the magnitude of the quantity in the place where the zero is located is nearer zero than any finite value then the zero is a significant number. In the number 0.00123 none of the

zeros is significant since they merely locate the decimal. As Fales[1] points out, the decimal point in any measurement is determined solely by the *unit* in which the measurement is expressed. We then can eliminate zeros which are not significant by simply using a unit which is ten, or a hundred, etc., times as large. Talbot's[2] excellent text states that the value 356000 inches contains no significant zeros. This is true according to the above statement; the value could be expressed as 356 units of 1000 inches each. Nevertheless, if the distance between two points is known to be nearer 356000 inches than 356001 inches or 355999 inches *then*[3] all the zeros in the number 356000 *are* significant. To circumvent all ambiguity one may choose the largest convenient unit and then use the zero only to fix the decimal. This is particularly simple to do when using metric units.

In general usage we have the following

RULES

I. As many digits should be used in any *observation* (i.e. any physical measurement) as will give only one uncertain or doubtful figure. In lengthy and very exact *calculations* it is sometimes desirable to retain, until the final result, two doubtful figures. This point will be discussed later.

II. In rejecting superfluous figures, (i.e. in "rounding off") increase by 1 the last figure retained if the following rejected figure is 5 or over. (Incidentally, all things considered, and especially if only few checks are made, this rule is more to be desired than another frequently followed, viz., if the discarded figure is a 5 either increase or decrease by 1 the last figure retained so that it becomes *even.*)

III. In general it is sufficient in addition and subtraction, multiplication and division, to say that the final result may not contain a greater number of significant figures than the least number of significant figures entering into the calculation. For example, as Talbot points out, the sum of the following three terms: 0.0121, 25.64 and 1.05788, assuming the last digit to be uncertain in each case, is

$$\begin{array}{r} 0.01 \\ 25.64 \\ 1.06 \\ \hline 26.71 \end{array}$$

[1] Fales, Inorganic Quantitative Analysis, p. 10. The Century Co. 1925.
[2] Talbot, Quantitative Chemical Analysis, p. 28. MacMillan Co., 1931.
[3] See also Smythe and Michels, Advanced Electrical Measurements, p. 7. Van Nostrand Co., 1932.

It is useless to extend the digits of any term further than the hundredths since the second term has its doubtful figure in that column. The third term is made 1.06 to conform with Rule II. That this third rule holds in this case is obvious if we assume that, had we been able to secure significant figures to the hundred thousandths place for all three numbers, they necessarily would have been respectively between those below, otherwise they would not round off to those given above.

Smallest possible		Largest possible	
0.01205		0.01214	
25.63500		25.64499	
1.05788		1.05788	
26.70493	Rounded off=26.70	26.71501	Rounded off=26.72

Therefore, if rounded off in the first place, before adding, we do not offend our precision since the value 26.71 is not in error by more than 0.01 when compared with the values 26.70 and 26.72 as obtained when the rounding off was deferred until the last step.

In regard to multiplication and division it is, of course, the *percentage* precision of the result which may not exceed the *percentage* precision of the least precise number entering the calculation. In general, but not always, this means that the number of significant figures in the factor of least precision will be identical with the number of digits in all other factors and in the result.[4]

IV. Regarding the use of logarithms, we retain as many significant figures in the mantissa as are retained in the factors themselves conforming to rule III. In addition it is well to point out that in the vast majority of chemical work, for example, the precision obtainable is limited to four significant figures. This is due, among other things, to the fact that atomic weights for so many of the elements (which appear so often in calculations) have only four figures given; with the ordinary analytical balance weighing is possible to the ten thousandths gram; a burette may ordinarily be read to the hundredths cubic centimeter,[5] etc. As a result a great many texts in chemistry include a *four*-place logarithm table. However, the writer is in complete agreement with Fales in preferring the five-place tables for chemical calculations. The reason for this preference is that it takes no more time to use the latter and, in order to secure four significant figures when employing the four-place tables, it is necessary to resort to the "proportional parts," whereas four figures are obtainable with the

[4] For a rigorous treatment see J. W. Mellor, Higher Mathematics for Students of Chemistry and Physics, p. 522. Longmans, Green & Co., 1909.

[5] Volumes are usually between ten and one hundred cc.

five-place tables without this time-consuming expediency. Furthermove, we are occasionally entitled to five significant figures in the result, and should, therefore, have the five-place habit.

It is sometimes confusing to the student when advised to use *two* uncertain figures in calculations for very exact work. A simple illustration in chemistry and physics will obviate the point. In the most accurate weighing with the ordinary analytical balance the last significant figure (the tenths milligram) is secured by the "method of swings." Without going into the details of the method itself it is sufficient to say that the whole process depends on obtaining the equilibrium or rest point of the oscillating pointer as it swings over a scale numbered, say, from 0 to 20 with 10 as the true equilibrium point for a perfectly adjusted balance. The actual equilibrium point is taken with the pans empty, and again with the load. The difference between the two is needed to establish the identity of the last significant figure. Suppose the readings of the pointer are as follows. (An even number of readings must be taken for swings in one direction and an odd number in the other.)

Readings, pans empty.

	left		right
	3.9		
	4.1		16.3
	4.3		16.1
mean=	4.10	and	16.20

$$\text{eq. pt.}=\frac{4.10+16.20}{2}=10.15,\ \text{Rounded off}=10.2$$

Readings with load.

	0.1		
	0.4		14.9
	0.8		14.4
mean=	0.43	and	14.65

$$\text{eq. pt.}=\frac{0.43+14.65}{2}=7.54,\ \text{Rounded off}=7.5$$

Now the actual readings can be made only to tenths of scale divisions; the tenths are estimated and are, therefore, doubtful. Nevertheless, if, let us say, these figures were obtained during the course of very exact work, it would be wise to retain two doubtful figures in

the *calculations*. For if, in seeking the difference in the two equilibrium points, we round off to tenths first we obtain: $10.2-7.5=2.7$. But if we subtract first and finally round off to tenths we obtain: $10.15-7.54=2.61$; rounded off$=2.6$. Thus a difference of 0.1 results if we fail to carry two doubtful figures throughout the calculations.

Another consideration which frequently escapes an otherwise cautious student is illustrated by the following example. Suppose the duplicate results of an experiment carried out with apparatus entitling one to four significant figures are found to be, say 78.12 and 78.18. One would conclude that 78.1 is certainly correct and that the mean value, 78.15 gave a good approximation for the hundredths place; it accordingly is proper to report this mean value with its four significant figures. However, had the results been 78.12 and 78.52 the deviation would be 0.40. In this case it is obvious that the *first* decimal place is only an approximation and the reported result should be 78.3 with only the three significant figures.

The writer is aware that his students fail to remember much which he would prefer they did not forget. And it is probably true that every teacher of mathematics includes these points in his elementary courses. Yet it is rare that a student has all of them at his command as he begins his study in the natural sciences. With the binomial theorem and other more abstruse concepts to master early in his study of algebra, no wonder the student is likely to neglect a proper appreciation of significant numbers. But it would be to his great advantage if, in his early courses of study in pure mathematics, he could be made surely to know that he will constantly be called upon to apply these principles herein outlined as he takes up experimental measurements.

Two more items the writer wishes to mention. The chemist and the physicist invariably place a zero before the decimal in expressing fractions, for example, 0.75 and not .75. Mistakes in copying are avoided by this custom. The European mathematician and the chemist of this country always express the natural logarithm by the symbol ln and the Briggian logarithm by the symbol log. It seems that these two customs might be employed with advantage by all engaged in natural sciences, pure or applied.

"The essence of science is tested imagination, qualitative ideas held down to quantitative measurements. As it was impossible for the American Indian to have any comprehensive and formulated knowledge of heat without thermometers, so is it impossible to have any science without measurements." —From N. J. Lennes, in *"A Survey Course in Mathematics."*

*SOME COMMENTS ON THE SECONDARY MATHEMATICS SITUATION

By C. H. SISAM

Colorado College

Mathematics has always been a basic element in education. For its tool value, for its potential vocational value, and for the training it affords in a particularly useful type of thinking, mathematics has been regularly placed as one of the fundamental subjects in any reasonable educational curriculum.

So essential, indeed, has the training in mathematics always been regarded that the subject is placed, with notable uniformity, as early in the curriculum as the student's mentality makes it apparently possible for him to assimilate it—to the end that those students whose formal training ended early might not have to go through life too seriously handicapped in this essential part of their education. I hold that it is not without significance that the last two years in the student's high school course—when he is mentally best prepared really to learn mathematics—are precisely the years in which he is most likely to be without any course whatever in the subject.

It seems absurd that the very importance of mathematics should be one of the chief contributing factors to the drive that some educators are making against mathematics and that its importance should be an essential element in the effectiveness of that drive, yet such I believe to be the case. We are told, not without a show of justice, that the mathematics we teach is too difficult for our students, that they do not really understand it or retain it, and that the time and energy it absorbs is disproportionate to the good that the student actually gets out of it. We are not told that these difficulties are largely because these very objectors have misplaced the subject in the curriculum.

We must agree, I think, that mathematics is difficult. The juvenile mind does not take kindly to its formal arguments and clear-cut distinctions and, the weaker the mentality of the victim, the less kindly does he take to the disentanglement of mathematical complications.

* Read before the Rocky Mountain Section of the National Mathematical Association of America, Golden, Colo., April 20, 1935.

Still worse, the organic structure of the subject puts under an increasing handicap on the weaker student who is trying to master its more advanced topics without being adequately prepared in its more elementary branches.

In recent decades, the high school mathematics teacher's problem has been further complicated by the fact that a much larger number of pupils of inferior mentality and, consequently, poorly assimilated elementary training, have been continuing their studies in the secondary schools, so that "educators" are increasingly saying: "Your results, dear high school mathematics teacher, are not equal to the time and effort that you demand of your students. Let us teach them, instead, dancing, basket weaving, or anything else that does not put too great a strain on their mentalities.

These are a few of the factors in the situation. Now, what can be done about it? Several things. Here are a few that can be, and, to a greater or less extent, are being done.

1. We can try to improve the teaching of high school mathematics. That there is room for improvement, no one will deny; but I believe that our high school teachers, taken as a group, are doing a hard job commendably well.

2. We can dilute the high school course to fit a lower student mentality. You may call this "soft pedagogy" or "facing the facts." Although everybody deplores it, it is being done. We must recognize, however, that we can not hope to compete, on the basis of easiness, with other subjects more favored by nature in this particular.

3. We can discourage the poorer students from taking secondary mathematics. This too often leads the short-sighted student into unforeseen difficulties; it often makes him feel that there is a premium on stupidity, and it deprives him of valuable training that we would like him to have.

4. Finally, we can delay the student's training in mathematics, throughout its entire course, for a year or two, to the end that his mental development may progress to a point where it will be somewhat easier for him to cope with the difficulties he is going to encounter.

It is this last device, which is doubtless the one least often actually employed, that I wish to emphasize here.

What I am trying to suggest, specifically, is something like this: that such mathematics as is taught in the first grade be moved to the second, that the second grade mathematics be transferred to the third, and so on; that eleventh grade mathematics be allocated to the twelfh grade and that twelfth grade mathematics be left to the colleges and universities. These changes should be made without increasing the difficulty and complexity of the work offered. Some rearrangements

will need to be made, but not in the direction of putting difficult things earlier.

I maintain that the men who oppose mathematics most vigorously are themselves, as curriculum makers, largely responsible for the shortcomings they rail against; since they, themselves, have been instrumental in permitting mathematics to remain psychologically misplaced in the course of study. They, themselves, have "consulted" too much the place where the student ought to learn his mathematics and too little his mental unpreparedness, at that age, to receive it profitably. I agree with the estimate of the value of mathematics as shown by its place in the curriculum; I agree, up to a point, with the estimate of the difficulty the student is having in mastering it. I disagree with the conclusion that this important subject, because of its difficulty, ought to be largely eliminated.

A couple of illustrations must suffice at this point. The subject of fractions is a bugbear to many pupils. These pupils do not understand fractions; they work with them mechanically and uncomprehendingly. Yet it is not particularly uncommon for these same students to gather, and to appreciate, the basic principle of fractions, a few years later, out of collateral topics, when their minds are better developed.

Again, Euclid's great textbook on geometry has stood as a monumental work for two thousand years and has dominated the teaching of the subject ever since it appeared. But Euclid's geometry was not written for tenth grade students. It is not adapted to their mentality; its logical form, remarkable as it seems to the adult mind, often repels the adolescent one. We have notably simplified the manner of presenting its content. Might we not advantageously teach this material at an age somewhat nearer to that at which it can best be appreciated?

From the college viewpoint, there would be a distinct advantage in shortening the interval between the finishing of high school mathematics and the beginning of college mathematics. Less opportunity to forget may in itself lower the mortality of students in first year college mathematics.

I recognize, and do not attempt to belittle the various curriculum changes that the proposed plan would necessitate. Some of these would involve much care and labor; some, taken by themselves, would doubtless be undesirable; others, I believe, would be distinct improvements. In any event, I raise only this question: would the benefits of this rearrangement, taken as a whole, outweigh its defects?

Problem Department

Edited by

T. A. BICKERSTAFF

This department aims to provide problems of varying degrees of difficulty which will interest anyone who is engaged in the study of mathematics.

All readers, whether subscribers or not, are invited to propose problems and to solve problems here proposed.

Problems and solutions will be credited to their authors.

While it is our aim to publish problems of most interest to the readers, it is believed that regular text-book problems are, as a rule, less interesting than others. Therefore, other problems will be given preference when the space for problems is limited.

Send all communications about problems to T. A Bickerstaff, University, Mississippi.

SOLUTIONS

No. 89. Proposed by W. V. Parker, Georgia School of Technology.

P is a fixed point on the ellipse $x^2/a^2+y^2/b^2=1$, and X and Y are two other points of the ellipse. Prove that the area of the triangle PXY is greatest when its centroid is at the center of the ellipse and that its greatest area is $3\sqrt{3}ab/4$.

Solution by *J. Rosenbaum, Hartford Federal College.*

Consider a plane M through the x axis and making with the xy plane the angle α whose cosine is b/a. It is well known that the ellipse of the problem is the projection on the xy plane of the circle C_1 in plane M, center at origin, and radius equal to a. Since the area of the projection on the xy plane of an area A in plane M is A cos α, it follows that if PXY is the projection of triangle P′X′Y′ in plane M, PXY will be a maximum when and only when P′X′Y′ is a maximum. But P′X′Y′is a maximum when it is equilateral, hence PXY is greatest when P′X′Y′ is equilateral. Finally, since the ratio of division of a line segment is not altered by this projection, a median of P′X′Y′ is projected into a median of PXY, and thus the projection of the centroid, O, of P′X′Y′ is the centroid of PXY. Since O is its own projection the first part of the problem is proved. To prove the area,

$$PXY = P'X'Y' \cos\alpha = (3\sqrt{3}a^2/4)b/a$$
$$= 3\sqrt{3}\ ab/4.$$

Also solved by R. A. Miller, A. C. Briggs, and the proposer.

No. 93. Proposed by Dewey C. Duncan, University of California.

All equilateral hyperbolas which pass through three fixed points of the plane intersect in a definite fourth point. What geometrical relationship exists among these four points?

Solution by *G. E. Raynor, Lehigh University.*

The four points form an ortho-centric group; that is, each one is the ortho-center of the triangle formed by the other three.

This well-known theorem can be proved elegantly by projective methods, but we give instead the following simple proof by elementary analytic Geometry.

The equation of any one of the hyperbolas can, by a suitable choice of axes, be written in the form

$$xy=a \tag{1}$$

Let three of the points be $P_1(x_1, y_1)$; $P_2(x_2, y_2)$; $P_3(x_3, y_3)$.

Then

$$x_1y_1=x_2y_2=x_3y_3=a. \tag{2}$$

The slope of P_1P_2 reduces by means of (2) to

$$-a/x_1x_2.$$

Hence the equation of the altitude of triangle $P_1P_2P_3$ which passes through P_3 is

$$y-y_3=x_1x_2/a\ (x-x_3), \tag{3}$$

and similarly of that through P_2 is

$$y-y_2=x_1x_3/a\ (x-x_2). \tag{4}$$

If we solve equations (3) and (4) for x and y making use of (2) we find for the coordinates of the ortho-center of $P_1P_2P_3$,

$$x=-a^2/x_1x_2x_3 \tag{5}$$

$$y=-x_1x_2x_3/a \tag{6}$$

It is seen at a glance that these values satisfy (1) and the theorem is proved.

It may be of interest to note that by considering merely the *signs* of x_1, x_2 and x_3 we have at once from (5) and (6) the following theorem.

If P_1, P_2 and P_3 all lie on the same branch of an equilateral hyperbola the ortho-center of triangle $P_1P_2P_3$ lies on the other branch;

while if only two of these lie on the same branch the ortho-center lies on that branch.

Also solved by Norman Anning, University of Michigan, and Karleton W. Crain, Purdue University.

PROBLEMS FOR SOLUTION

No. 99. Proposed by Nathan Altshiller-Court, University of Oklahoma.

Prove that the sum of the edges of a parallelopiped is smaller than the sum and greater than half the sum of the diagonals of its faces.

No. 100. Proposed by W. V. Parker, Georgia Tech.

If P_1, P_2, P_3, P_4 are the incenter and the three excenters of a triangle, the system of conics on them is a system of equilateral hyperbolas with centers on the circumscribing circle of the triangle.

No. 101. Proposed by Nathan Altshiller-Court.

Construct a sphere passing through a given point so that the distances of its center from three given points shall be equal respectively to a Pth, Qth, and Rth part of the radius.

No. 102. Proposed by Norman Anning, University of Michigan.

Prove that any one of the points (lm, o), (mn, o), (o, nl), $(o, -m^2)$ is the ortho-center of the triangle which has the other points for vertices.

No. 103. Proposed by Nathan Altshiller-Court.

The sum of the powers of a point with respect to four given spheres is equal to two-thirds of the sum of the powers of the same points with respect to the six radical spheres* of the given spheres taken two at a time.

LATE SOLUTIONS

No. 88. By W. V. Parker.

"By a combination of physical, mathematical and philosophic thought genius has made it possible to answer, by means of exact methods, questions concerning the universe which seemed doomed forever to remain the objects of vain speculation." —From Schlick's *"Space and Time."*

* Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p 173, MacMillan, 1935.

Book Reviews

Edited by

P. K. Smith

Plane Trigonometry. By H. L. Reitz, J. F. Reilly, and Roscoe Woods. The MacMillan Company, New York, 1935-X+142pp.

This text was written largely around the idea of presenting the subject matter in the least confusing and the simplest manner. The definitions of the sine, cosine, and tangent for an acute angle are first given. Following some exercises on the three principal functions the three remaining functions are defined for an acute angle. The functions of the general angle are not defined until the fundamental ideas on the functions of an acute angle are well established by exercises and problems. Following the first thirty pages devoted to the functions of an acute angle are Chapters III, IV, and V which are devoted respectively to functions of any angle, identities, and graphical representations of the functions.

In Chapter VI the law of sines and the law of cosines, and applications are taken up. Logarithms are not used in this chapter. Chapter VII contains the addition theorems with their related formulas, and more identities. Logarithms are treated in Chapter VIII. A complete treatment of the use of the tables for the functions is given in this chapter. The solutions of oblique triangle with logarithms is covered in Chapter IX. The last three Chapters, X, XI and XII, are devoted respectively to radian measures, inverse functions, and trigonometric equations.

This text contains seventy-two pages of four and five place tables, answers to odd problems and an index. This is a splendid text in plane trigonometry. There is nothing original or new in the course, but for one desiring a change of text book this book is worthy of one's examination.

P. K. Smith

A First Course in Calculus. By Herman L. Slobin and Marvin R. Solt. Farrar and Rinehart, New York, 1935-XI+426pp.

The reviewer will begin with the first sentence in the preface of the text "The Authors of A First Course in Calculus hope that they have succeeded in presenting the elements of theory and the applications of the calculus in as simple a form as possible and with

as much rigor in demonstrations as is necessary and possible in a first course." The authors have succeeded in carrying out their main objectives. The text is written in a forceful and effective style. Definitions and fundamental concepts are treated in a spirit of rigor and with much care. In Chapter I the usual topics, viz., constants, variables, functions and limits are treated. The authors give very interesting examples and exercises on limits and continuous functions.

Chapter II treats the fundamental concept of the derivative. This short chapter is well done. The authors are very careful in defining a tangent to a curve. If a student works each one of the twenty-four well-chosen exercises in Chapter II, he will be ready for Chapter III which treats of the general rules of differentiation. Chapter IV covers rules for differentiations of the transcendental functions as well as the definitions and exercises on the differential of a function. Chapter V contains chiefly problems on maxima and minima. In Chapter VI higher derivatives and radius of curvature are taken up.

"The Application of the Derivative to Physical Problems" is treated in Chapter VII. The subject of rates is not treated up to this point. The chapter begins with rectilinear motions, then, comes curvilinear motions, and finally the general rate problems enter. It is certainly wise to postpone rate problems until the student is well grounded in calculus.

The next two chapters are devoted to "undetermined" forms and partial differentiation. The study of these two topics might be deferred to some advantage until later in the course. The definite integral and the formal rules of integration are covered in Chapters X, XI and XII.

"The Integral as a Limit of a Sum" constitutes the subject matter of Chapter XIII. The illustrations leading up to the concept of the fundamental theorem of the calculus are somewhat prolix. Some of this space could have been devoted to a treatment of Duhamel's theorem. With the use of this powerful theorem the authors could have avoided the idea of "dropping infinitesimals of higher order."

In Chapter XIV multiple integrals are treated, and in Chapter XV centroids, pressure, moment of inertia and potential are taken up. In this last chapter the authors lay their foundations well for a thorough mastery. The last three chapters, XVI, XVII and XVIII, are devoted respectively to infinite series, expansion of a function, and to differential equations.

The text is worthy of close scrutiny with a view to adoption. The figures are splendid and the printing is well done.

P. K. SMITH